

Unified Description of Matrix Mechanics and Wave Mechanics on Hydrogen Atom

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Abstract

A new mathematical method is established to represent the operator, wave functions and square matrix in the same representation. We can obtain the specific square matrices corresponding to the angular momentum and Runge-Lenz vector operators with invoking assistance from the operator relations and the orthonormal wave functions. Furthermore, the first-order differential equations will be given to deduce the specific wave functions without using the solution of the second order Schrödinger equation. As a result, we will unify the descriptions of the matrix mechanics and the wave mechanics on hydrogen atom. By using matrix transformations, we will also deduce the specific matrix representations of the operators in the SO(4,2) group.

Keywords: Matrix Operator Wave function Hydrogen atom Quantum mechanics

1. Introduction

In 1925, based on Niels Bohr's correspondence principle, Werner Heisenberg represented the spatial coordinate q and the momentum p by the following form ^[1]

$$q = [q(nm)e^{2\pi i\nu(nm)t}], p = [p(nm)e^{2\pi i\nu(nm)t}] \quad (1)$$

Max Born and Pascual Jordan then wrote q substituted for $q(nm)$ as a matrix [2]

$$\begin{bmatrix} 0 & q(01) & 0 & 0 & 0 & \dots \\ q(10) & 0 & q(12) & 0 & 0 & \dots \\ 0 & q(21) & 0 & q(23) & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

They abandoned the representation (1) in favor of the shorter notation

$$q = q(nm) \quad p = p(nm)$$

The founders of matrix mechanics tried to describe the mechanics quantum by the square matrix. They had not been successful because the source of the matrix could not be explained. In modern quantum mechanics the mechanical quantities was described by the operator. However, the operator was studied in isolation without being related to the wave functions so that the square matrix in quantum mechanics looked very mysterious. In fact, the square matrix is derived from the superposition coefficient.

To address this issue, we offer the following new approaches.

In linear algebra, the product of row matrix and the same order column matrix is equal to a polynomial. For example

$$\begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = A_1 B_1 + A_2 B_2$$

Now that equality holds from left to right, the equality should hold from right to left

$$A_1 B_1 + A_2 B_2 = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad (2)$$

So a polynomial can be expanded to the product of row matrix and the same order column matrix. Therefore

I, The following expressions are equivalent

$$\begin{bmatrix} a\psi_1 + b\psi_2 & c\psi_1 + d\psi_2 \end{bmatrix} = \begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

and

$$\begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a\psi_1 + b\psi_2 & c\psi_1 + d\psi_2 \end{bmatrix}$$

II, If

$$\begin{cases} \hat{A}\psi_1 = a\psi_1 + b\psi_2 \\ \hat{A}\psi_2 = c\psi_1 + d\psi_2 \end{cases} \quad (3)$$

then

$$\hat{A} \begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} = \begin{bmatrix} a\psi_1 + b\psi_2 & c\psi_1 + d\psi_2 \end{bmatrix} = \begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \quad (4)$$

III, If

$$\begin{cases} \hat{B}\psi_1 = e\psi_1 + f\psi_2 \\ \hat{B}\psi_2 = g\psi_1 + h\psi_2 \end{cases} \quad (5)$$

then

$$\hat{B} \begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} = \begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} \begin{bmatrix} e & g \\ f & h \end{bmatrix} \quad (6)$$

Thus, from (4) and (6), we get

$$\hat{A}\hat{B} \begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} = \hat{A} \begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} \begin{bmatrix} e & g \\ f & h \end{bmatrix} = \begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} e & g \\ f & h \end{bmatrix} \quad (7)$$

In fact, we are also able to get from (3) and (5)

$$\hat{A}\hat{B} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} e\hat{A}\psi_1 + f\hat{A}\psi_2 \\ g\hat{A}\psi_1 + h\hat{A}\psi_2 \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} \hat{A}\psi_1 \\ \hat{A}\psi_2 \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

The expression (7) is clearer than the above expression, so (3) is represented as (4) and we adopt (7) in this article.

For one-dimensional harmonic oscillator, the results of wave mechanics are

$$E_1 = \frac{1}{2}\hbar\omega, E_2 = \frac{3}{2}\hbar\omega, \dots, E_n = (n - \frac{1}{2})\hbar\omega, \quad (8)$$

$$\psi_1 = (\frac{\alpha^2}{\pi})^{\frac{1}{4}} e^{-\frac{\alpha^2}{2}x^2}, \dots, \psi_n = \frac{H_{n-1}(\alpha x)}{2^{\frac{n-1}{2}}[(n-1)!]^{\frac{1}{2}}} (\frac{\alpha^2}{\pi})^{\frac{1}{4}} e^{-\frac{\alpha^2}{2}x^2} (\alpha = \sqrt{\frac{\mu\omega}{\hbar}}) \quad (9)$$

with

$$H_{n-1}(z) = (-1)^{n-1} e^{z^2} \frac{d^{n-1}}{dz^{n-1}} e^{(-z^2)} (z = \alpha x)$$

Thus

$$\hat{x} \begin{bmatrix} \psi_1 & \psi_2 & \dots & \psi_{n-1} & \psi_n \end{bmatrix} = \begin{bmatrix} \psi_1 & \psi_2 & \dots & \psi_{n-1} & \psi_n \end{bmatrix}$$

$$\frac{1}{\sqrt{2}\alpha} \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \sqrt{n} \\ 0 & 0 & \cdots & \sqrt{n} & 0 \end{bmatrix} \quad (10)$$

$$\hat{p} \begin{bmatrix} \psi_1 & \psi_2 & \cdots & \psi_{n-1} & \psi_n \end{bmatrix} = \begin{bmatrix} \psi_1 & \psi_2 & \cdots & \psi_{n-1} & \psi_n \end{bmatrix} i\hbar \frac{\alpha}{\sqrt{2}} \begin{bmatrix} 0 & -1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & -\sqrt{n} \\ 0 & 0 & \cdots & \sqrt{n} & 0 \end{bmatrix} \quad (11)$$

In classic group theory, the following results can be obtained by using the raising and lowering operator method.

$$\hat{a}\psi_1 = 0, \hat{a}\psi_2 = \psi_1, \cdots, \hat{a}\psi_n = \sqrt{n-1}\psi_{n-1} \quad (12)$$

$$\psi_2 = \hat{a}^+\psi_1, \psi_3 = \frac{1}{\sqrt{2}}\hat{a}^+\psi_2, \cdots, \psi_n = \frac{1}{\sqrt{n-1}}\hat{a}^+\psi_{n-1}, \hat{a}^+\psi_n = 0 \quad (13)$$

with

$$\hat{a} = \frac{1}{\sqrt{2}\alpha}(\alpha^2\hat{x} + \frac{i}{\hbar}\hat{p}), \hat{a}^+ = \frac{1}{\sqrt{2}\alpha}(\alpha^2\hat{x} - \frac{i}{\hbar}\hat{p})$$

The square matrices in (10) and (11) can be deduced from (12) and (13).

In fact, contrary to the above method, we assume that in terms of the theorem in the text

$$\hat{H}\psi_1 = E_1\psi_1, \hat{H}\psi_2 = E_2\psi_2, \cdots, \hat{H}\psi_n = E_n\psi_n \quad (14)$$

$$\hat{x} \begin{bmatrix} \psi_1 & \psi_2 & \cdots & \psi_n \end{bmatrix} = \begin{bmatrix} \psi_1 & \psi_2 & \cdots & \psi_n \end{bmatrix} \begin{bmatrix} X_{11} & X_{21}^* & \cdots & X_{n1}^* \\ X_{21} & X_{22} & \cdots & X_{n2}^* \\ \cdots & \cdots & \cdots & \cdots \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{bmatrix} \quad (15)$$

where $X_{11}, X_{22}, \cdots, X_{nn}$ are real numbers. According to (4)-(7), we can convert the following operator relations into the square matrix relations with invoking assistance from (14) and (15).

$$[\hat{x}, \hat{H}] = i\frac{\hbar}{\mu}\hat{p}$$

$$[\hat{p}, \hat{H}] = -i\hbar\mu\omega^2\hat{x}$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$\hat{H} = \frac{\hat{p}^2}{2\mu} + \frac{1}{2}\mu\omega^2\hat{x}^2$$

The matrix element in (10) and (11) and (8) can be deduced from those square matrix relations. Furthermore, the first order differential equations are got from (10) and (11). As a result, the solutions of these equations are just (9).

In the classic quantum mechanics, the operator of Runge-Lenz vector ^[3] is defined as

$$\vec{\hat{A}} = \frac{1}{\mu}(\hat{\vec{p}} \times \vec{\hat{L}} - i\hbar\hat{\vec{p}}) - k\frac{\vec{r}}{r} \quad (16)$$

We replace $\vec{\hat{A}}$ by ^[4]

$$\vec{\hat{A}'} = \sqrt{-\frac{\mu}{2\hat{H}}}\vec{\hat{A}}$$

and introduce two new operators

$$\vec{\hat{Q}} = \frac{1}{2}(\vec{\hat{L}} + \vec{\hat{A}'}), \vec{\hat{Q}'} = \frac{1}{2}(\vec{\hat{L}} - \vec{\hat{A}'}) \quad (17)$$

Each $\vec{\hat{Q}}$ and $\vec{\hat{Q}'}$ constitutes a closed SU(2) Lie algebra and they generate the group SO(4,2). With invoking assistance from the Wigner-Eckart theorem, the relevant SO(4) representations have been built up to determine the matrix elements of the group generators. However, as we know, the square matrix in (4) depends on the arrangement of the wave functions in the row matrix. The matrix elements don't represent the specific matrix.

The new method will be applied to obtain the square matrices of the angular momentum and Runge-Lenz vector operators corresponding to the wave functions which are the orthonormalized simultaneous eigenfunctions of the operators \hat{L}_z, \hat{L}^2 and \hat{H} . By using matrix transformations, we will transform these matrices to the square matrices responding to the wave functions which are the orthonormalized simultaneous eigenfunctions of the operators \hat{Q}_z, \hat{Q}^2 and \hat{H} .

In order to unify the descriptions of the matrix mechanics and the wave mechanics on hydrogen atom, we need to establish the corresponding mathematical foundations.

2. Mathematical foundations

In quantum mechanics, the wave function describes quantum state. Therefore, two definitions on wave function are given as follows

Definition 1 If $\phi = \sum_{I=1}^n c_I \psi_I$, then Dirac's ket and bra vector are written as $|\phi\rangle = \sum_{I=1}^n c_I |\psi_I\rangle$ and $\langle\phi| = \sum_{I=1}^n c_I^* \langle\psi_I|$ respectively, where $*$ denotes complex conjugation.

Definition 2 If ϕ and φ are two arbitrary wave functions, then the inner product of ϕ and φ is $\langle\phi|\varphi\rangle = \int \phi^* \varphi d\tau$.

Let $\psi_1, \psi_2, \dots, \psi_n$ be the orthonormalized wave functions. We have in terms of definition 2:

$$\begin{bmatrix} \langle\psi_1| \\ \langle\psi_2| \\ \dots \\ \langle\psi_n| \end{bmatrix} \begin{bmatrix} |\psi_1\rangle & |\psi_2\rangle & \dots & |\psi_n\rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix} \quad (18)$$

According to Born's probability interpretation and matrix transformation, we can always assume that $\psi_1, \psi_2, \dots, \psi_n$ are a set of orthonormalized wave functions. Simultaneously

$$\hat{F}\psi_I = F_I\psi_I (I = 1, 2, \dots, n)$$

where F_1, F_2, \dots, F_n are called eigenvalues of the operator \hat{F} corresponding to the eigenfunctions.

When an operator \hat{A} acts on the orthonormalized wave functions $\psi_1, \psi_2, \dots, \psi_n$, we will obtain some new wave functions $\phi_1 = \hat{A}\psi_1, \phi_2 = \hat{A}\psi_2, \dots, \phi_n = \hat{A}\psi_n$. According to the principle of superposition states in quantum mechanics, these new wave functions $\phi_1, \phi_2, \dots, \phi_n$ were represented by linear combination of orthonormalized wave functions $\psi_1, \psi_2, \dots, \psi_n$.

$$\hat{A}\psi_1 = \phi_1 = \sum_{I=1}^n A_{I1}\psi_I, \hat{A}\psi_2 = \phi_2 = \sum_{I=1}^n A_{I2}\psi_I, \dots, \hat{A}\psi_n = \phi_n = \sum_{I=1}^n A_{In}\psi_I \quad (19)$$

According to definition 1, the expression (19) can be written as

$$|\hat{A}\psi_1\rangle = \sum_{I=1}^n A_{I1}|\psi_I\rangle, |\hat{A}\psi_2\rangle = \sum_{I=1}^n A_{I2}|\psi_I\rangle, \dots, |\hat{A}\psi_n\rangle = \sum_{I=1}^n A_{In}|\psi_I\rangle$$

It follows that in terms of the inverse law of matrix multiplication

$$[|\hat{A}\psi_1\rangle |\hat{A}\psi_2\rangle \cdots |\hat{A}\psi_n\rangle] = [|\psi_1\rangle |\psi_2\rangle \cdots |\psi_n\rangle] \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

Both sides of the above expression are left multiplied by the column matrix $[\langle \psi_1 | \langle \psi_2 | \cdots \langle \psi_n |]^T$. Combining with (18), we get

$$\begin{bmatrix} \langle \psi_1 | \hat{A} \psi_1 \rangle & \langle \psi_1 | \hat{A} \psi_2 \rangle & \cdots & \langle \psi_1 | \hat{A} \psi_n \rangle \\ \langle \psi_2 | \hat{A} \psi_1 \rangle & \langle \psi_2 | \hat{A} \psi_2 \rangle & \cdots & \langle \psi_2 | \hat{A} \psi_n \rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle \psi_n | \hat{A} \psi_1 \rangle & \langle \psi_n | \hat{A} \psi_2 \rangle & \cdots & \langle \psi_n | \hat{A} \psi_n \rangle \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

If \hat{A} is a Hermitian operator then for two arbitrary wave functions ϕ and φ , we have

$$\langle \phi | \varphi \rangle = \langle \varphi | \phi \rangle^*$$

Hence

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} = \begin{bmatrix} A_{11}^* & A_{21}^* & \cdots & A_{n1}^* \\ A_{21}^* & A_{22}^* & \cdots & A_{n2}^* \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1}^* & A_{n2}^* & \cdots & A_{nn}^* \end{bmatrix}$$

Therefore, we get

Theorem: If \hat{A} is a Hermitian operator, then $\hat{A}\phi_K = \sum_{I=1}^n A_{IK}\psi_I$ ($K = 1, 2, \cdots, n$) in the orthonormalized complete function sets $\phi_1, \psi_2, \cdots, \psi_n$ are written as

$$\hat{A} \begin{bmatrix} \psi_1 & \psi_2 & \cdots & \psi_n \end{bmatrix} = \begin{bmatrix} \psi_1 & \psi_2 & \cdots & \psi_n \end{bmatrix} \begin{bmatrix} A_{11} & A_{21}^* & \cdots & A_{n1}^* \\ A_{21} & A_{22} & \cdots & A_{n2}^* \\ \cdots & \cdots & \cdots & \cdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \quad (20)$$

where $A_{11}, A_{22}, \cdots, A_{nn}$ are real numbers.

In quantum mechanics, all of the operators that describe mechanical quantities are Hermitian operators, so we can deal with the degenerate hydrogen atom by means of the above theorem.

3. Angular Momentum

In the spherical coordinate, the angular momentum operator is given by

$$\begin{aligned}\vec{\hat{L}} &= \hat{L}_x \vec{i} + \hat{L}_y \vec{j} + \hat{L}_z \vec{k} \\ \left\{ \begin{array}{l} \hat{L}_x = i\hbar(\sin\varphi \frac{\partial}{\partial\theta} + \cot\theta \cos\varphi \frac{\partial}{\partial\varphi}) \\ \hat{L}_y = i\hbar(-\cos\varphi \frac{\partial}{\partial\theta} + \cot\theta \sin\varphi \frac{\partial}{\partial\varphi}) \\ \hat{L}_z = -i\hbar \frac{\partial}{\partial\varphi} \end{array} \right. \end{aligned} \quad (21)$$

It is well known that the commutation relations between the components of the angular momentum operators are written as

$$[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x \quad (22)$$

$$[\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y \quad (23)$$

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z \quad (24)$$

$$[\hat{L}^2, \hat{L}_x] = [\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z] = 0 \quad (25)$$

Thus, let L_1, L_2, \dots, L_s be the eigenvalue of the operator \hat{L}_z and Y_1, Y_2, \dots, Y_s be the orthonormalized simultaneous eigenfunctions of the operators \hat{L}_z and \hat{L}^2 respectively, then

$$\hat{L}_z \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_s \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_s \end{bmatrix} \begin{bmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & L_s \end{bmatrix} \quad (26)$$

Because \hat{L}_x is a Hermitian operator, it is assumed in terms of the theorem that

$$\hat{L}_x \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_s \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_s \end{bmatrix} \begin{bmatrix} L_{11} & L_{21}^* & \cdots & L_{s1}^* \\ L_{21} & L_{22} & \cdots & L_{s2}^* \\ \cdots & \cdots & \cdots & \cdots \\ L_{s1} & L_{s2} & \cdots & L_{ss} \end{bmatrix} \quad (27)$$

where $L_{11}, L_{22}, \dots, L_{ss}$ are real numbers. From (23), (26) and (27), we get

$$\hat{L}_y \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_s \end{bmatrix} = \frac{1}{i\hbar} (\hat{L}_z \hat{L}_x - \hat{L}_x \hat{L}_z) \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_s \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_s \end{bmatrix}$$

$$\left(-\frac{i}{\hbar}\right) \begin{bmatrix} 0 & (L_1 - L_2)L_{21}^* & \cdots & (L_1 - L_s)L_{s1}^* \\ (L_2 - L_1)L_{21} & 0 & \cdots & (L_2 - L_s)L_{s2}^* \\ \cdots & \cdots & \cdots & \cdots \\ (L_s - L_1)L_{s1} & (L_s - L_2)L_{s2} & \cdots & 0 \end{bmatrix} \quad (28)$$

Combining with (22) and (26), we have

$$\hat{L}_x \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_s \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_s \end{bmatrix} \quad (29)$$

$$\frac{1}{\hbar^2} \begin{bmatrix} 0 & (L_2 - L_1)^2 L_{21}^* & \cdots & (L_s - L_1)^2 L_{s1}^* \\ (L_2 - L_1)^2 L_{21} & 0 & \cdots & (L_s - L_2)^2 L_{s2}^* \\ \cdots & \cdots & \cdots & \cdots \\ (L_s - L_1)^2 L_{s1} & (L_s - L_2)^2 L_{s2} & \cdots & 0 \end{bmatrix}$$

Because $L_1, L_2, L_3, \dots, L_{s-1}, L_s$ are real numbers and it is clear that $L_s > L_{s-1} > \dots > L_3 > L_2 > L_1$, comparison of (27) with (29) yields

$$\begin{aligned} L_{11} &= L_{22} = \cdots = L_{ss} = 0 \\ (L_2 - L_1)^2 L_{21} &= \hbar^2 L_{21} \Rightarrow L_2 - L_1 = \hbar \\ (L_3 - L_1)^2 L_{31} &= \hbar^2 L_{31}, (L_3 - L_2)^2 L_{32} = \hbar^2 L_{32} \Rightarrow L_3 - L_2 = \hbar, L_{31} = 0 \\ &\cdots \\ (L_s - L_1)^2 L_{s1} &= \hbar^2 L_{s1}, \dots, (L_s - L_{s-1})^2 L_{ss-1} = \hbar^2 L_{ss-1} \Rightarrow L_s - L_{s-1} = \hbar, \\ L_{s1} &= \cdots = L_{ss-2} = 0 \end{aligned}$$

Therefore

$$L_2 = L_1 + \hbar, L_3 = L_1 + 2\hbar, \dots, L_s = L_1 + (s-1)\hbar \quad (30)$$

And that (27) and (28) become

$$\begin{aligned} &\hat{L}_x \begin{bmatrix} Y_1 & Y_2 & Y_3 & \cdots & Y_{s-2} & Y_{s-1} & Y_s \end{bmatrix} \\ &= \begin{bmatrix} Y_1 & Y_2 & Y_3 & \cdots & Y_{s-2} & Y_{s-1} & Y_s \end{bmatrix} \\ &\begin{bmatrix} 0 & L_{21}^* & 0 & \cdots & 0 & 0 & 0 \\ L_{21} & 0 & L_{32}^* & \cdots & 0 & 0 & 0 \\ 0 & L_{32} & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & L_{s-1s-2}^* & 0 \\ 0 & 0 & 0 & \cdots & L_{s-1s-2} & 0 & L_{ss-1}^* \\ 0 & 0 & 0 & \cdots & 0 & L_{ss-1} & 0 \end{bmatrix} \end{aligned} \quad (31)$$

$$\begin{aligned}
& \hat{L}_y \begin{bmatrix} Y_1 & Y_2 & Y_3 & \cdots & Y_{s-2} & Y_{s-1} & Y_s \end{bmatrix} \\
& = \begin{bmatrix} Y_1 & Y_2 & Y_3 & \cdots & Y_{s-2} & Y_{s-1} & Y_s \end{bmatrix} \\
& (-i) \begin{bmatrix} 0 & -L_{21}^* & 0 & \cdots & 0 & 0 & 0 \\ L_{21} & 0 & -L_{32}^* & \cdots & 0 & 0 & 0 \\ 0 & L_{32} & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & -L_{s-1s-2}^* & 0 \\ 0 & 0 & 0 & \cdots & L_{s-1s-2} & 0 & -L_{ss-1}^* \\ 0 & 0 & 0 & \cdots & 0 & L_{ss-1} & 0 \end{bmatrix} \quad (32)
\end{aligned}$$

From (24), (31) and (32), we get

$$\begin{aligned}
& \hat{L}_z \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_{s-1} & Y_s \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_{s-1} & Y_s \end{bmatrix} \\
& \frac{2}{\hbar} \begin{bmatrix} -|L_{21}|^2 & 0 & \cdots & 0 & 0 \\ 0 & |L_{21}|^2 - |L_{32}|^2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & |L_{s-1s-2}|^2 - |L_{ss-1}|^2 & 0 \\ 0 & 0 & \cdots & 0 & |L_{ss-1}|^2 \end{bmatrix} \quad (33)
\end{aligned}$$

Comparison of (33) with (26) yields

$$|L_{21}|^2 = -\frac{\hbar}{2}L_1, |L_{32}|^2 = -\frac{\hbar}{2}(L_1 + L_2), \cdots, |L_{ss-1}|^2 = -\frac{\hbar}{2}(L_1 + L_2 + \cdots + L_{s-1}) \quad (34)$$

and $|L_{ss-1}|^2 = \frac{\hbar}{2}L_s$. Hence, $L_1 + L_2 + \cdots + L_s = 0$. Combining with (30), we have

$$L_1 = -j\hbar(j = \frac{s-1}{2}) \quad (35)$$

From (30) and (35), we get

$$L_1 = -j\hbar, L_2 = (1-j)\hbar, \cdots, L_{2j+1} = j\hbar \quad (36)$$

Thus, (26) become

$$\hat{L}_z \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_{2j+1} \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_{2j+1} \end{bmatrix} \hbar \begin{bmatrix} -j & 0 & \cdots & 0 \\ 0 & 1-j & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & j \end{bmatrix} \quad (37)$$

From (34) and (36), we get

$$|L_{21}|^2 = \frac{j}{2}\hbar^2, |L_{32}|^2 = \frac{2j-1}{2}\hbar^2, \cdots, |L_{2j+12j}|^2 = \frac{j}{2}\hbar^2 \quad (38)$$

From (31)-(32) and (36)-(38), we have

$$\hat{L}^2 \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_{2j+1} \end{bmatrix} = j(j+1)\hbar^2 \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_{2j+1} \end{bmatrix} \quad (39)$$

If the wave functions $Y_1, Y_2, \dots, Y_{2j+1}$ is relabeled as $Y_{j-j}, Y_{j1-j}, \dots, Y_{jj}$, then (37) and (39) can be written as

$$\hat{L}_z Y_{jm} = m\hbar Y_{jm} (m = -j, 1-j, \dots, j) \quad (40)$$

$$\hat{L}^2 Y_{jm} = j(j+1)\hbar^2 Y_{jm} (m = -j, 1-j, \dots, j) \quad (41)$$

Let $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$ and $\hat{L}_- = \hat{L}_x - i\hat{L}_y$, we can get from (31) and (32)

$$\begin{aligned} \hat{L}_+ \begin{bmatrix} Y_{j-j} & Y_{j1-j} & \cdots & Y_{jj-1} & Y_{jj} \end{bmatrix} &= \begin{bmatrix} Y_{j-j} & Y_{j1-j} & \cdots & Y_{jj-1} & Y_{jj} \end{bmatrix} \\ &2 \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ L_{21} & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & L_{2j+12j} & 0 \end{bmatrix} \end{aligned} \quad (42)$$

$$\begin{aligned} \hat{L}_- \begin{bmatrix} Y_{j-j} & Y_{j1-j} & \cdots & Y_{jj-1} & Y_{jj} \end{bmatrix} &= \begin{bmatrix} Y_{j-j} & Y_{j1-j} & \cdots & Y_{jj-1} & Y_{jj} \end{bmatrix} \\ &2 \begin{bmatrix} 0 & L_{21}^* & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & L_{2j+12j}^* \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \end{aligned} \quad (43)$$

In fact, from (30), (34) and (35), we get

$$|L_{p+1p}|^2 = -\frac{\hbar}{2}(L_1 + L_2 + \cdots + L_p) = \frac{\hbar^2}{4}p(2j-p+1) (p = 1, 2, \dots, 2j)$$

When $p = j + m + 1$

$$|L_{j+m+2j+m+1}|^2 = \frac{\hbar^2}{4}(j+m+1)(j-m) (m = -j, 1-j, \dots, j-1) \quad (44)$$

If we take positive real solutions from (44), then (42) become

$$\hat{L}_+ Y_{jm} = \hbar \sqrt{(j+m+1)(j-m)} Y_{jm+1} (m = -j, 1-j, \dots, j) \quad (45)$$

We can also find (45) from (40)-(43) with invoking assistance from

$$\hat{L}_- \hat{L}_+ = \hat{L}^2 - \hat{L}_z^2 - \hbar \hat{L}_z$$

When $p = j + m$

$$|L_{j+1+mj+m}|^2 = \frac{\hbar^2}{4}(j+m)(j+1-m)(m=1-j, \dots, j-1, j) \quad (46)$$

If we take positive real solutions from (46), then (43) become

$$\hat{L}_- Y_{jm} = \hbar \sqrt{(j+m)(j+1-m)} Y_{j, m-1} (m = -j, 1-j, \dots, j) \quad (47)$$

We can also find (47) from (40)-(43) with invoking assistance from

$$\hat{L}_+ \hat{L}_- = \hat{L}^2 - \hat{L}_z^2 + \hbar \hat{L}_z$$

I If s is an odd number, then

1) When $j=0$, (40)-(41), (45) and (47) are respectively written as

$$\hat{L}_+ Y_{00} = \hat{L}_- Y_{00} = \hat{L}_z Y_{00} = \hat{L}^2 Y_{00} = 0 \quad (48)$$

2) When $j=1$, (40)-(41), (45) and (47) are respectively written as

$$\hat{L}_+ Y_{1m_3} = \hbar \sqrt{(2+m_3)(1-m_3)} Y_{1m_3+1} \quad (49)$$

$$\hat{L}_- Y_{1m_3} = \hbar \sqrt{(1+m_3)(2-m_3)} Y_{1m_3-1} \quad (50)$$

$$\hat{L}_z Y_{1m_3} = m_3 \hbar Y_{1m_3} \quad (51)$$

$$\hat{L}^2 Y_{1m_3} = 2\hbar^2 Y_{1m_3} \quad (52)$$

with

$$m_3 = -1, 0, 1$$

3) When $j=2$, (40)-(41), (45) and (47) are respectively written as

$$\hat{L}_+ Y_{2m_5} = \hbar \sqrt{(3+m_5)(2-m_5)} Y_{2m_5+1} \quad (53)$$

$$\hat{L}_- Y_{2m_5} = \hbar \sqrt{(2+m_5)(3-m_5)} Y_{2m_5-1} \quad (54)$$

$$\hat{L}_z Y_{2m_5} = m_5 \hbar Y_{2m_5} \quad (55)$$

$$\hat{L}^2 Y_{2m_5} = 6\hbar^2 Y_{2m_5} \quad (56)$$

with

$$m_5 = -2, -1, 0, 1, 2$$

...

4) When $j = l$, (40)-(41), (45) and (47) are respectively written as

$$\hat{L}_+ Y_{lm_{2l+1}} = \hbar \sqrt{(l+1+m_{2l+1})(l-m_{2l+1})} Y_{lm_{2l+1}+1} \quad (57)$$

$$\hat{L}_- Y_{lm_{2l+1}} = \hbar \sqrt{(l+m_{2l+1})(l+1-m_{2l+1})} Y_{lm_{2l+1}-1} \quad (58)$$

$$\hat{L}_z Y_{lm_{2l+1}} = m_{2l+1} \hbar Y_{lm_{2l+1}} \quad (59)$$

$$\hat{L}^2 Y_{lm_{2l+1}} = l(l+1) \hbar^2 Y_{lm_{2l+1}} \quad (60)$$

with

$$m_{2l+1} = -l, 1-l, \dots, l$$

II: If s is an even number, then

1) When $j = \frac{1}{2}$, (40)-(41), (45) and (47) are respectively written as

$$\hat{L}_+ Y_{\frac{1}{2}m_2} = \hbar \sqrt{(\frac{3}{2}+m_2)(\frac{1}{2}-m_2)} Y_{\frac{1}{2}m_2+1} \quad (61)$$

$$\hat{L}_- Y_{\frac{1}{2}m_2} = \hbar \sqrt{(\frac{1}{2}+m_2)(\frac{3}{2}-m_2)} Y_{\frac{1}{2}m_2-1} \quad (62)$$

$$\hat{L}_z Y_{\frac{1}{2}m_2} = m_2 \hbar Y_{\frac{1}{2}m_2} \quad (63)$$

$$\hat{L}^2 Y_{\frac{1}{2}m_2} = \frac{3}{4} \hbar^2 Y_{\frac{1}{2}m_2} \quad (64)$$

with

$$m_2 = -\frac{1}{2}, \frac{1}{2}$$

...

2) When $j = l - \frac{1}{2}$, the equations (40)-(41), (45) and (47) are respectively written as

$$\hat{L}_+ Y_{l-\frac{1}{2}m_{2l}} = \hbar \sqrt{(l+\frac{1}{2}+m_{2l})(l-\frac{1}{2}-m_{2l})} Y_{l-\frac{1}{2}m_{2l}+1} \quad (65)$$

$$\hat{L}_- Y_{l-\frac{1}{2}m_{2l}} = \hbar \sqrt{(l-\frac{1}{2}+m_{2l})(l+\frac{1}{2}-m_{2l})} Y_{l-\frac{1}{2}m_{2l}-1} \quad (66)$$

$$\hat{L}_z Y_{l-\frac{1}{2}m_{2l}} = m_{2l} \hbar Y_{l-\frac{1}{2}m_{2l}} \quad (67)$$

$$\hat{L}^2 Y_{l-\frac{1}{2}m_{2l}} = (l - \frac{1}{2})(l + \frac{1}{2}) \hbar^2 Y_{l-\frac{1}{2}m_{2l}} \quad (68)$$

with

$$m_{2l} = \frac{1}{2} - l, \dots, l - \frac{1}{2}$$

4. The Energy Levels of Hydrogen Atom

The Hamiltonian operator of hydrogen atom is written as:

$$\hat{H} = -\frac{\hbar^2}{2\mu} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) + \frac{\hat{L}^2}{2\mu r^2} - \frac{k}{r} \quad (69)$$

In the spherical coordinate, The operator of Runge-Lenz vector is given by

$$\begin{aligned} \vec{\hat{A}} &= \hat{A}_x \vec{i} + \hat{A}_y \vec{j} + \hat{A}_z \vec{k} \\ \left\{ \begin{array}{l} \hat{A}_x = \frac{1}{\mu} (\hat{p}_y \hat{L}_z - \hat{p}_z \hat{L}_y - i \hbar \hat{p}_x) - k \sin \theta \cos \varphi \\ \hat{A}_y = \frac{1}{\mu} (\hat{p}_z \hat{L}_x - \hat{p}_x \hat{L}_z - i \hbar \hat{p}_y) - k \sin \theta \sin \varphi \\ \hat{A}_z = \frac{1}{\mu} (\hat{p}_x \hat{L}_y - \hat{p}_y \hat{L}_x - i \hbar \hat{p}_z) - k \cos \theta \end{array} \right. \end{aligned} \quad (70)$$

Let $\hat{A}_+ = \hat{A}_x + i \hat{A}_y$ and $\hat{A}_- = \hat{A}_x - i \hat{A}_y$, we can show that

$$[\hat{A}_z, \hat{L}_+] = \hbar \hat{A}_+ \quad (71)$$

$$[\hat{L}_-, \hat{A}_z] = \hbar \hat{A}_- \quad (72)$$

$$[\hat{A}_+, \hat{L}_-] = 2\hbar \hat{A}_z \quad (73)$$

$$[\hat{L}_z, \hat{A}_z] = 0 \quad (74)$$

$$\hat{A}_+ \hat{L}_- + \hat{A}_- \hat{L}_+ + 2\hat{A}_z \hat{L}_z = 0 \quad (75)$$

$$\hat{A}_- \hat{A}_+ + \hat{A}_z^2 = \frac{2\hat{H}}{\mu} (\hat{L}^2 + \hbar \hat{L}_z + \hbar^2) + k^2 \quad (76)$$

and

$$\begin{aligned} [\hat{L}_z, \hat{A}_+] &= \hbar \hat{A}_+, [\hat{A}_-, \hat{L}_z] = \hbar \hat{A}_-, [\hat{L}_+, \hat{A}_-] = 2\hbar \hat{A}_z \\ [\hat{L}_+, \hat{A}_+] &= [\hat{L}_-, \hat{A}_-] = 0 \end{aligned}$$

$$[\hat{A}_+, \hat{A}_-] = -\frac{4\hbar}{\mu} \hat{H} \hat{L}_z, [\hat{A}_+, \hat{A}_z] = \frac{2\hbar}{\mu} \hat{H} \hat{L}_+, [\hat{A}_z, \hat{A}_-] = \frac{2\hbar}{\mu} \hat{H} \hat{L}_- \\ [\hat{A}_+, \hat{H}] = [\hat{A}_-, \hat{H}] = [\hat{A}_z, \hat{H}] = 0$$

It is well known that

$$[\hat{L}_z, \hat{H}] = [\hat{L}^2, \hat{H}] = [\hat{L}_z, \hat{L}^2] = 0$$

Hence, the operators \hat{H} , \hat{L}_z and \hat{L}^2 have simultaneous eigenfunctions.

When N=1

Let $\psi_{100} = R_{10}Y_{00}$ be an normalized simultaneous eigenfunction of the operators \hat{H} , \hat{L}_z and \hat{L}^2 . It is assumed that

$$\hat{A}_z \psi_{100} = A \psi_{100} \quad (77)$$

Combining with (48) and (71), we have

$$\hat{A}_+ \psi_{100} = 0 \quad (78)$$

From (48), (72) and (77), we get:

$$\hat{A}_- \psi_{100} = 0 \quad (79)$$

From (48), (73) and (78), we get:

$$\hat{A}_z \psi_{100} = 0 \quad (80)$$

And that from (48), (76) and (78)-(80) we get:

$$E_1 = -\frac{\mu k^2}{2\hbar^2} \quad (81)$$

When n=2:

Let $\psi_{200} = R_{20}Y_{00}$, $\psi_{21-1} = R_{21}Y_{1-1}$, $\psi_{210} = R_{21}Y_{10}$ and $\psi_{211} = R_{21}Y_{11}$ be the orthonormalized simultaneous eigenfunctions of the operators \hat{H} , \hat{L}_z and \hat{L}^2 . From (48)-(52), we get

$$\hat{L}_+ \begin{bmatrix} \psi_{200} & \psi_{21-1} & \psi_{210} & \psi_{211} \end{bmatrix} \\ = \begin{bmatrix} \psi_{200} & \psi_{21-1} & \psi_{210} & \psi_{211} \end{bmatrix} \sqrt{2}\hbar \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (82)$$

$$\begin{aligned}
& \hat{L}_- [\psi_{200} \quad \psi_{21-1} \quad \psi_{210} \quad \psi_{211}] \\
&= [\psi_{200} \quad \psi_{21-1} \quad \psi_{210} \quad \psi_{211}] \sqrt{2}\hbar \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{83}$$

$$\begin{aligned}
& \hat{L}_z [\psi_{200} \quad \psi_{21-1} \quad \psi_{210} \quad \psi_{211}] \\
&= [\psi_{200} \quad \psi_{21-1} \quad \psi_{210} \quad \psi_{211}] \hbar \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{84}$$

$$\begin{aligned}
& \hat{L}^2 [\psi_{200} \quad \psi_{21-1} \quad \psi_{210} \quad \psi_{211}] \\
&= [\psi_{200} \quad \psi_{21-1} \quad \psi_{210} \quad \psi_{211}] \hbar^2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}
\end{aligned} \tag{85}$$

Because \hat{A}_z is a Hermitian operator, it is assumed in terms of the theorem that:

$$\begin{aligned}
& \hat{A}_z [\psi_{200} \quad \psi_{21-1} \quad \psi_{210} \quad \psi_{211}] = [\psi_{200} \quad \psi_{21-1} \quad \psi_{210} \quad \psi_{211}] \\
& \begin{bmatrix} A_{11}(2) & A_{21}^*(2) & A_{31}^*(2) & A_{41}^*(2) \\ A_{21}(2) & A_{22}(2) & A_{32}^*(2) & A_{42}^*(2) \\ A_{31}(2) & A_{32}(2) & A_{33}(2) & A_{43}^*(2) \\ A_{41}(2) & A_{42}(2) & A_{43}(2) & A_{44}(2) \end{bmatrix}
\end{aligned} \tag{86}$$

where $A_{11}(2)$, $A_{22}(2)$, $A_{33}(2)$ and $A_{44}(2)$ are real numbers. Combining with (74) and (84), we have:

$$\begin{aligned}
& \hat{A}_z [\psi_{200} \quad \psi_{21-1} \quad \psi_{210} \quad \psi_{211}] = [\psi_{200} \quad \psi_{21-1} \quad \psi_{210} \quad \psi_{211}] \\
& \begin{bmatrix} A_{11}(2) & 0 & A_{31}^*(2) & 0 \\ 0 & A_{22}(2) & 0 & 0 \\ A_{31}(2) & 0 & A_{33}(2) & 0 \\ 0 & 0 & 0 & A_{44}(2) \end{bmatrix}
\end{aligned} \tag{87}$$

Combining with (71) and (82), we get

$$\hat{A}_+ [\psi_{200} \quad \psi_{21-1} \quad \psi_{210} \quad \psi_{211}] = [\psi_{200} \quad \psi_{21-1} \quad \psi_{210} \quad \psi_{211}]$$

$$\sqrt{2} \begin{bmatrix} 0 & A_{31}^*(2) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & A_{33}(2) - A_{22}(2) & 0 & 0 \\ -A_{31}(2) & 0 & A_{44}(2) - A_{33}(2) & 0 \end{bmatrix} \quad (88)$$

From (73), (83), (87) and (88), we get

$$A_{33}(2) = A_{11}(2) = 0, A_{22}(2) = -A_{44}(2) \quad (89)$$

Combining with (72), (83) and (87), we get

$$\begin{aligned} \hat{A}_- [\psi_{200} \quad \psi_{21-1} \quad \psi_{210} \quad \psi_{211}] &= [\psi_{200} \quad \psi_{21-1} \quad \psi_{210} \quad \psi_{211}] \\ \sqrt{2} \begin{bmatrix} 0 & 0 & 0 & -A_{31}^*(2) \\ A_{31}(2) & 0 & A_{44}(2) & 0 \\ 0 & 0 & 0 & A_{44}(2) \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (90)$$

From (75), (82)-(84) and (87)-(90) we get

$$A_{44}(2) = 0 \quad (91)$$

Therefore, (87), (88) and (90) become respectively

$$\begin{aligned} \hat{A}_+ [\psi_{200} \quad \psi_{21-1} \quad \psi_{210} \quad \psi_{211}] &= [\psi_{200} \quad \psi_{21-1} \quad \psi_{210} \quad \psi_{211}] \\ \sqrt{2} \begin{bmatrix} 0 & A_{31}^*(2) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -A_{31}(2) & 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (92)$$

$$\begin{aligned} \hat{A}_- [\psi_{200} \quad \psi_{21-1} \quad \psi_{210} \quad \psi_{211}] &= [\psi_{200} \quad \psi_{21-1} \quad \psi_{210} \quad \psi_{211}] \\ \sqrt{2} \begin{bmatrix} 0 & 0 & 0 & -A_{31}^*(2) \\ A_{31}(2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (93)$$

$$\begin{aligned} \hat{A}_z [\psi_{200} \quad \psi_{21-1} \quad \psi_{210} \quad \psi_{211}] &= [\psi_{200} \quad \psi_{21-1} \quad \psi_{210} \quad \psi_{211}] \\ \begin{bmatrix} 0 & 0 & A_{31}^*(2) & 0 \\ 0 & 0 & 0 & 0 \\ A_{31}(2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (94)$$

From (76), (84)-(85) and (92)-(94), we get

$$E_2 = -\frac{\mu k^2}{2\hbar^2} \frac{1}{2^2} \quad (95)$$

$$|A_{31}(2)|^2 = \frac{k^2}{4} \quad (96)$$

If we take positive real solutions from (96), then (92)-(94) become

$$\begin{cases} \hat{A}_+ \psi_{200} = -\frac{\sqrt{2}}{2} k \psi_{211} \\ \hat{A}_+ \psi_{21-1} = \frac{\sqrt{2}}{2} k \psi_{200}, \hat{A}_+ \psi_{210} = 0, \hat{A}_+ \psi_{211} = 0 \end{cases} \quad (97)$$

$$\begin{cases} \hat{A}_- \psi_{200} = \frac{\sqrt{2}}{2} k \psi_{21-1} \\ \hat{A}_- \psi_{21-1} = 0, \hat{A}_- \psi_{210} = 0, \hat{A}_- \psi_{211} = -\frac{\sqrt{2}}{2} k \psi_{200} \end{cases} \quad (98)$$

$$\begin{cases} \hat{A}_z \psi_{200} = \frac{k}{2} \psi_{210} \\ \hat{A}_z \psi_{21-1} = 0, \hat{A}_z \psi_{210} = \frac{k}{2} \psi_{200}, \hat{A}_z \psi_{211} = 0 \end{cases} \quad (99)$$

When n=3

Let $\psi_{300} = R_{30}Y_{00}, \psi_{31-1} = R_{31}Y_{1-1}, \psi_{310} = R_{31}Y_{10}, \psi_{311} = R_{31}Y_{11}, \psi_{32-2} = R_{32}Y_{2-2}, \psi_{32-1} = R_{32}Y_{2-1}, \psi_{320} = R_{32}Y_{20}, \psi_{321} = R_{32}Y_{21}$ and $\psi_{322} = R_{32}Y_{22}$ be the orthonormalized simultaneous eigenfunctions of the operators \hat{H}, \hat{L}_z and \hat{L}^2 . From (48)-(56), we get

$$\begin{aligned} & \hat{L}_+ [\psi_{300} \quad \psi_{31-1} \quad \psi_{310} \quad \psi_{311} \quad \psi_{322} \quad \psi_{321} \quad \psi_{320} \quad \psi_{32-1} \quad \psi_{32-2}] \\ &= [\psi_{300} \quad \psi_{31-1} \quad \psi_{310} \quad \psi_{311} \quad \psi_{322} \quad \psi_{321} \quad \psi_{320} \quad \psi_{32-1} \quad \psi_{32-2}] \\ & \quad \hbar \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (100) \\ & \hat{L}_- [\psi_{300} \quad \psi_{31-1} \quad \psi_{310} \quad \psi_{311} \quad \psi_{322} \quad \psi_{321} \quad \psi_{320} \quad \psi_{32-1} \quad \psi_{32-2}] \end{aligned}$$

$$\begin{aligned}
&= [\psi_{300} \quad \psi_{31-1} \quad \psi_{310} \quad \psi_{311} \quad \psi_{322} \quad \psi_{321} \quad \psi_{320} \quad \psi_{32-1} \quad \psi_{32-2}] \\
&\quad \hbar \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix} \quad (101)
\end{aligned}$$

$$\begin{aligned}
&\hat{L}_z [\psi_{300} \quad \psi_{31-1} \quad \psi_{310} \quad \psi_{311} \quad \psi_{322} \quad \psi_{321} \quad \psi_{320} \quad \psi_{32-1} \quad \psi_{32-2}] \\
&= [\psi_{300} \quad \psi_{31-1} \quad \psi_{310} \quad \psi_{311} \quad \psi_{322} \quad \psi_{321} \quad \psi_{320} \quad \psi_{32-1} \quad \psi_{32-2}] \\
&\quad \hbar \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix} \quad (102)
\end{aligned}$$

$$\begin{aligned}
&\hat{L}^2 [\psi_{300} \quad \psi_{31-1} \quad \psi_{310} \quad \psi_{311} \quad \psi_{322} \quad \psi_{321} \quad \psi_{320} \quad \psi_{32-1} \quad \psi_{32-2}] \\
&= [\psi_{300} \quad \psi_{31-1} \quad \psi_{310} \quad \psi_{311} \quad \psi_{322} \quad \psi_{321} \quad \psi_{320} \quad \psi_{32-1} \quad \psi_{32-2}] \\
&\quad \hbar^2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix} \quad (103)
\end{aligned}$$

Because \hat{A}_z is a Hermitian operator, it is assumed in terms of the theorem that

$$\begin{aligned}
& \hat{A}_z \begin{bmatrix} \psi_{300} & \psi_{31-1} & \psi_{310} & \psi_{311} & \psi_{322} & \psi_{321} & \psi_{320} & \psi_{32-1} & \psi_{32-2} \end{bmatrix} \\
&= \begin{bmatrix} \psi_{300} & \psi_{31-1} & \psi_{310} & \psi_{311} & \psi_{322} & \psi_{321} & \psi_{320} & \psi_{32-1} & \psi_{32-2} \end{bmatrix} \\
& \begin{bmatrix} A_{11}(3) A_{21}^*(3) A_{31}^*(3) A_{41}^*(3) A_{51}^*(3) A_{61}^*(3) A_{71}^*(3) A_{81}^*(3) A_{91}^*(3) \\ A_{21}(3) A_{22}(3) A_{32}^*(3) A_{42}^*(3) A_{52}^*(3) A_{62}^*(3) A_{72}^*(3) A_{82}^*(3) A_{92}^*(3) \\ A_{31}(3) A_{32}(3) A_{33}(3) A_{43}^*(3) A_{53}^*(3) A_{63}^*(3) A_{73}^*(3) A_{83}^*(3) A_{93}^*(3) \\ A_{41}(3) A_{42}(3) A_{43}(3) A_{44}(3) A_{54}^*(3) A_{64}^*(3) A_{74}^*(3) A_{84}^*(3) A_{94}^*(3) \\ A_{51}(3) A_{52}(3) A_{53}(3) A_{54}(3) A_{55}(3) A_{65}^*(3) A_{75}^*(3) A_{85}^*(3) A_{95}^*(3) \\ A_{61}(3) A_{62}(3) A_{63}(3) A_{64}(3) A_{65}(3) A_{66}(3) A_{76}^*(3) A_{86}^*(3) A_{96}^*(3) \\ A_{71}(3) A_{72}(3) A_{73}(3) A_{74}(3) A_{75}(3) A_{76}(3) A_{77}(3) A_{87}^*(3) A_{97}^*(3) \\ A_{81}(3) A_{82}(3) A_{83}(3) A_{84}(3) A_{85}(3) A_{86}(3) A_{87}(3) A_{88}(3) A_{98}^*(3) \\ A_{91}(3) A_{92}(3) A_{93}(3) A_{94}(3) A_{95}(3) A_{96}(3) A_{97}(3) A_{98}(3) A_{99}(3) \end{bmatrix} \quad (104)
\end{aligned}$$

where $A_{11}(3), A_{22}(3), \dots, A_{88}(3)$ and $A_{99}(3)$ are real numbers. From (71)-(75), (100)-(102) and (104), we get

$$\begin{aligned}
& \hat{A}_+ \begin{bmatrix} \psi_{300} & \psi_{31-1} & \psi_{310} & \psi_{311} & \psi_{322} & \psi_{321} & \psi_{320} & \psi_{32-1} & \psi_{32-2} \end{bmatrix} \\
&= \begin{bmatrix} \psi_{300} & \psi_{31-1} & \psi_{310} & \psi_{311} & \psi_{322} & \psi_{321} & \psi_{320} & \psi_{32-1} & \psi_{32-2} \end{bmatrix} \\
& \begin{bmatrix} 0 & \sqrt{2}A_{31}^*(3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2A_{82}^*(3) \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}A_{82}^*(3) & 0 & 0 \\ -\sqrt{2}A_{31}(3) & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{3}A_{82}^*(3) & 0 & 0 \\ 0 & 0 & 0 & -2A_{82}(3) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2}A_{82}(3) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{6}}{3}A_{82}(3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (105)
\end{aligned}$$

$$\begin{aligned}
& \hat{A}_- \begin{bmatrix} \psi_{300} & \psi_{31-1} & \psi_{310} & \psi_{311} & \psi_{322} & \psi_{321} & \psi_{320} & \psi_{32-1} & \psi_{32-2} \end{bmatrix} \\
&= \begin{bmatrix} \psi_{300} & \psi_{31-1} & \psi_{310} & \psi_{311} & \psi_{322} & \psi_{321} & \psi_{320} & \psi_{32-1} & \psi_{32-2} \end{bmatrix}
\end{aligned}$$

$$\begin{bmatrix}
0 & 0 & 0 & -\sqrt{2}A_{31}^*(3) & 0 & 0 & 0 & 0 & 0 \\
\sqrt{2}A_{31}(3) & 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{6}}{3}A_{82}^*(3) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{2}A_{82}^*(3) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2A_{82}^*(3) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\sqrt{6}}{3}A_{82}(3) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{2}A_{82}(3) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2A_{82}(3) & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \quad (106)$$

$$\begin{aligned}
& \hat{A}_z \begin{bmatrix} \psi_{300} & \psi_{31-1} & \psi_{310} & \psi_{311} & \psi_{322} & \psi_{321} & \psi_{320} & \psi_{32-1} & \psi_{32-2} \end{bmatrix} \\
& = \begin{bmatrix} \psi_{300} & \psi_{31-1} & \psi_{310} & \psi_{311} & \psi_{322} & \psi_{321} & \psi_{320} & \psi_{32-1} & \psi_{32-2} \end{bmatrix} \\
& \begin{bmatrix}
0 & 0 & A_{31}^*(3) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{82}^*(3) & 0 \\
A_{31}(3) & 0 & 0 & 0 & 0 & 0 & \frac{2}{3}\sqrt{3}A_{82}^*(3) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A_{82}^*(3) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A_{82}(3) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2}{3}\sqrt{3}A_{82}(3) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A_{82}(3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \quad (107)
\end{aligned}$$

with

$$\frac{A_{64}(3)}{\sqrt{3}} = \frac{A_{73}(3)}{2} = \frac{A_{82}(3)}{\sqrt{3}} \quad (108)$$

From (76), (102)-(103) and (105)-(107), we get

$$E_3 = -\frac{\mu k^2}{2\hbar^2} \frac{1}{3^2} \quad (109)$$

$$|A_{82}(3)|^2 = \frac{k^2}{9}, |A_{31}(3)|^2 = \frac{8k^2}{27} \quad (110)$$

If we take positive real solutions from (110), then (105)-(107) become

$$\begin{cases} \hat{A}_+ \psi_{300} = -\frac{4\sqrt{3}}{9}k\psi_{311}; \hat{A}_+ \psi_{31-1} = \frac{4\sqrt{3}}{9}k\psi_{300} - \frac{\sqrt{6}}{9}k\psi_{320}, \\ \hat{A}_+ \psi_{310} = -\frac{\sqrt{2}}{3}k\psi_{321}, \hat{A}_+ \psi_{311} = -\frac{2}{3}k\psi_{322}; \hat{A}_+ \psi_{322} = 0, \hat{A}_+ \psi_{321} = 0, \\ \hat{A}_+ \psi_{320} = \frac{\sqrt{6}}{9}k\psi_{311}, \hat{A}_+ \psi_{32-1} = \frac{\sqrt{2}}{3}k\psi_{310}, \hat{A}_+ \psi_{32-2} = \frac{2}{3}k\psi_{31-1} \end{cases} \quad (111)$$

$$\begin{cases} \hat{A}_- \psi_{300} = \frac{4\sqrt{3}}{9}k\psi_{31-1}; \hat{A}_- \psi_{31-1} = \frac{2}{3}k\psi_{32-2}, \hat{A}_- \psi_{310} = \frac{\sqrt{2}}{3}k\psi_{32-1}, \\ \hat{A}_- \psi_{311} = \frac{\sqrt{6}}{9}k\psi_{320} - \frac{4\sqrt{3}}{9}k\psi_{300}; \hat{A}_- \psi_{322} = -\frac{2}{3}k\psi_{32-2}, \\ \hat{A}_- \psi_{321} = -\frac{\sqrt{2}}{3}k\psi_{310}, \hat{A}_- \psi_{320} = -\frac{\sqrt{6}}{9}k\psi_{31-1}, \hat{A}_- \psi_{32-1} = 0, \hat{A}_- \psi_{32-2} = 0 \end{cases} \quad (112)$$

$$\begin{cases} \hat{A}_z \psi_{300} = \frac{2\sqrt{6}}{9} k \psi_{310}; \hat{A}_z \psi_{31-1} = \frac{k}{3} \psi_{32-1}, \hat{A}_z \psi_{310} = \frac{2\sqrt{6}}{9} k \psi_{300} + \frac{2\sqrt{3}}{9} k \psi_{320}, \\ \hat{A}_z \psi_{311} = \frac{k}{3} \psi_{321}; \hat{A}_z \psi_{322} = 0, \hat{A}_z \psi_{321} = \frac{k}{3} k \psi_{311}, \\ \hat{A}_z \psi_{320} = \frac{2\sqrt{3}}{9} k \psi_{310}, \hat{A}_z \psi_{32-1} = \frac{k}{3} \psi_{31-1}, \hat{A}_z \psi_{32-2} = 0 \end{cases} \quad (113)$$

...

When $N = n$ (Let n be the even number)

Let $\psi_{n00} = R_{n0}Y_{00}, \dots, \psi_{nl-l} = R_{nl}Y_{l-l}, \dots, \psi_{nll-1} = R_{nl}Y_{l-1}$ and $\psi_{nll} = R_{nl}Y_{ll}$ be the orthonormalized simultaneous eigenfunctions of the operators \hat{H}, \hat{L}_z and \hat{L}^2 . Because \hat{A}_z is a Hermitian operator, it is assumed in terms of the theorem that

$$\begin{aligned} & \hat{A}_z [\psi_{n00} \quad \dots \quad \psi_{nl-2l-2} \quad \dots \quad \psi_{nl-11-l} \quad \dots \quad \psi_{nll}] \\ &= [\psi_{n00} \quad \dots \quad \psi_{nl-2l-2} \quad \dots \quad \psi_{nl-11-l} \quad \dots \quad \psi_{nll}] \\ & \begin{bmatrix} A_{11}(n) & \dots & A_{(l-1)^2 1}^*(n) & \dots & A_{l^2 1}^*(n) & \dots & A_{(l+1)^2 1}^*(n) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ A_{(l-1)^2 1}(n) & \dots & A_{(l-1)^2 (l-1)^2}(n) & \dots & A_{l^2 (l-1)^2}^*(n) & \dots & A_{(l+1)^2 (l-1)^2}^*(n) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ A_{l^2 1}(n) & \dots & A_{l^2 (l-1)^2}(n) & \dots & A_{l^2 l^2}(n) & \dots & A_{(l+1)^2 l^2}^*(n) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ A_{(l+1)^2 1}(n) & \dots & A_{(l+1)^2 (l-1)^2}(n) & \dots & A_{(l+1)^2 l^2}(n) & \dots & A_{(l+1)^2 (l+1)^2}(n) \end{bmatrix} \end{aligned} \quad (114)$$

where $A_{11}(n), \dots, A_{(l-1)^2 (l-1)^2}(n), \dots, A_{l^2 l^2}(n), \dots, A_{(l+1)^2-1(l+1)^2-1}(n)$ and $A_{(l+1)^2 (l+1)^2}(n)$ are real numbers. From (48)-(51), (53)-(55), \dots , (57)-(59) (71)-(75) and (114), we get

$$\begin{aligned} & \hat{A}_+ [\dots \psi_{nl-2l-2} \quad \psi_{nl-1l-1} \quad \psi_{nl-1l-2} \quad \psi_{nl-1l-3} \quad \dots \quad \psi_{nl-11-l} \quad \psi_{nl-l} \quad \psi_{nl1-l} \quad \psi_{nl2-l} \quad \dots \quad \psi_{nll-2} \quad \psi_{nll-1} \quad \psi_{nll}] \\ &= [\dots \psi_{nl-2l-2} \quad \psi_{nl-1l-1} \quad \psi_{nl-1l-2} \quad \psi_{nl-1l-3} \quad \dots \quad \psi_{nl-11-l} \quad \psi_{nl-l} \quad \psi_{nl1-l} \quad \psi_{nl2-l} \quad \dots \quad \psi_{nll-2} \quad \psi_{nll-1} \quad \psi_{nll}] \\ & \begin{bmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & 0 & 0 & \sqrt{\frac{2}{2l^2-3}} T_{n-1}^* & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \dots & -\sqrt{2(l-1)} T_{n-1} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \sqrt{\frac{2}{2l^2-1}} T_n^* & 0 & 0 & 0 \\ \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & 0 & 0 & 0 & \dots & 0 & \sqrt{2l} T_n^* & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & 0 & 0 & 0 & 0 & \dots & -\sqrt{\frac{2}{2l^2-1}} T_n & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & 0 & 0 & -\sqrt{\frac{(2l-2)(2l-3)}{2l^2-1}} T_n & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & 0 & 0 & -\sqrt{2l-2} T_n & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & 0 & -\sqrt{2l} T_n & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (115)$$

$$\begin{aligned} & \hat{A}_- [\dots \psi_{nl-2l-2} \quad \psi_{nl-1l-1} \quad \psi_{nl-1l-2} \quad \psi_{nl-1l-3} \quad \dots \quad \psi_{nl-11-l} \quad \psi_{nl-l} \quad \psi_{nl1-l} \quad \psi_{nl2-l} \quad \dots \quad \psi_{nll-2} \quad \psi_{nll-1} \quad \psi_{nll}] \\ &= [\dots \psi_{nl-2l-2} \quad \psi_{nl-1l-1} \quad \psi_{nl-1l-2} \quad \psi_{nl-1l-3} \quad \dots \quad \psi_{nl-11-l} \quad \psi_{nl-l} \quad \psi_{nl1-l} \quad \psi_{nl2-l} \quad \dots \quad \psi_{nll-2} \quad \psi_{nll-1} \quad \psi_{nll}] \end{aligned}$$

$$\begin{bmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & 0 & -\sqrt{2(l-1)}T_{n-1}^* & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & \sqrt{\frac{2}{2l-3}}T_{n-1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -\sqrt{\frac{(2l-2)(2l-3)}{2l-1}}T_n^* \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -\sqrt{\frac{2}{2l-1}}T_n^* & \cdots & 0 \\
\cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\cdots & 0 & \sqrt{\frac{2}{2l-1}}T_n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix} \quad (116)$$

$$\begin{aligned}
\hat{A}_z [\cdots \psi_{nl-22-l} \cdots \psi_{nl-2l-2} \psi_{nl-1l-1} \psi_{nl-1l-2} \cdots \psi_{nl-12-l} \psi_{nl-11-l} \psi_{nl-l} \psi_{nl1-l} \cdots \psi_{nl1-1} \psi_{nl}] \\
= [\cdots \psi_{nl-22-l} \cdots \psi_{nl-2l-2} \psi_{nl-1l-1} \psi_{nl-1l-2} \cdots \psi_{nl-12-l} \psi_{nl-11-l} \psi_{nl-l} \psi_{nl1-l} \cdots \psi_{nl1-1} \psi_{nl}]
\end{aligned}$$

$$\begin{bmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & T_{n-1}^* & 0 & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & 0 & \cdots & 0 & 0 & T_{n-1}^* & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & T_n^* \\
\cdots & 0 & \cdots & T_{n-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & T_{n-1} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & T_n^* & \cdots & 0 \\
\cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & T_n & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & 0 & \cdots & 0 & T_n & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix} \quad (117)$$

with

$$T_n = A_{(l+1)^2-1(l-1)^2+1}(n), T_{n-1} = A_{l^2-1(l-2)^2+1}(n), \cdots, T_2 = A_{31}(n)$$

$$\left\{ \begin{aligned} \frac{A_{(l+1)^2-1(l-1)^2+1}(n)}{\sqrt{1(2l-1)}} &= \frac{A_{(l+1)^2-2(l-1)^2+2}(n)}{\sqrt{2(2l-2)}} = \cdots = \frac{A_{(l+1)^2-(2l-1)(l-1)^2+(2l-1)}(n)}{\sqrt{(2l-1)[2l-(2l-1)]}} \\ \frac{A_{l^2-1(l-2)^2+1}(n)}{\sqrt{1[(2l-2)-1]}} &= \frac{A_{l^2-2(l-2)^2+2}(n)}{\sqrt{2[(2l-2)-2]}} = \cdots = \frac{A_{l^2-(2l-3)(l-2)^2+(2l-3)}(n)}{\sqrt{(2l-3)[(2l-2)-(2l-3)]}} \\ &\cdots \\ \frac{A_{82}(n)}{\sqrt{1(4-1)}} &= \frac{A_{73}(n)}{\sqrt{2(4-2)}} = \frac{A_{64}(n)}{\sqrt{3(4-3)}} \end{aligned} \right. \quad (118)$$

From (48), (51)-(52), (55)-(56), \cdots (59)-(60), (76) and (115)-(117), we get

$$E_n = -\frac{\mu k^2}{2\hbar^2} \frac{1}{n^2} (n = l+1) \quad (119)$$

$$\begin{cases} |A_{(l+1)^2-1(l-1)^2+1}(n)|^2 = \frac{n^2-l^2}{4l^2-1} \frac{k^2}{n^2} (2l-1) \\ |A_{l^2-1(l-2)^2+1}(n)|^2 = \frac{n^2-(l-1)^2}{4(l-1)^2-1} \frac{k^2}{n^2} (2l-3) \\ \dots \\ |A_{31}(n)|^2 = \frac{n^2-1}{3} \frac{k^2}{n^2} \end{cases} \quad (120)$$

If we take positive real solutions from (120), then

$$\begin{aligned} \hat{A}_+ \psi_{njm_{2j+1}} &= \frac{k}{n} \sqrt{\frac{n^2-j^2}{4j^2-1}} (j-m_{2j+1})(j-m_{2j+1}-1) \psi_{nj-1m_{2j+1}+1} \\ &- \frac{k}{n} \sqrt{\frac{n^2-(j+1)^2}{4(j+1)^2-1}} (j+m_{2j+1}+1)(j+m_{2j+1}+2) \psi_{nj+1m_{2j+1}+1} \end{aligned} \quad (121)$$

$$\begin{aligned} \hat{A}_- \psi_{njm_{2j+1}} &= \frac{k}{n} \sqrt{\frac{n^2-(j+1)^2}{4(j+1)^2-1}} (j-m_{2j+1}+1)(j-m_{2j+1}+2) \psi_{nj+1m_{2j+1}-1} \\ &- \frac{k}{n} \sqrt{\frac{n^2-j^2}{4j^2-1}} (j+m_{2j+1})(j+m_{2j+1}-1) \psi_{nj-1m_{2j+1}-1} \end{aligned} \quad (122)$$

$$\begin{aligned} \hat{A}_z \psi_{njm_{2j+1}} &= \frac{k}{n} \sqrt{\frac{n^2-j^2}{4j^2-1}} (j^2-m_{2j+1}^2) \psi_{nj-1m_{2j+1}} \\ &+ \frac{k}{n} \sqrt{\frac{n^2-(j+1)^2}{4(j+1)^2-1}} [(j+1)^2-m_{2j+1}^2] \psi_{nj+1m_{2j+1}} \end{aligned} \quad (123)$$

with

$$\begin{cases} j=0; m_1=0 \\ j=1; m_3=-1, 0, 1 \\ \dots \\ j=l; m_{2l+1}=-l, 1-l, \dots, l \end{cases}$$

5. Discussion

1) When $j=0$, from (48), we get

$$\hat{L}_z Y_{00} = 0, \hat{L}_+ Y_{00} = 0 \quad (124)$$

2) When $j=1$, from (49)-(51), we get

$$\hat{L}_z Y_{11} = \hbar Y_{11}, \hat{L}_+ Y_{11} = 0; \hat{L}_- Y_{11} = \sqrt{2}\hbar Y_{10}, \hat{L}_- Y_{10} = \sqrt{2}\hbar Y_{1-1} \quad (125)$$

3) When $j = 2$, from (53)-(55), we get

$$\begin{cases} \hat{L}_z Y_{22} = 2\hbar Y_{22}, \hat{L}_+ Y_{22} = 0; \hat{L}_- Y_{22} = 2\hbar Y_{21}, \\ \hat{L}_- Y_{21} = \sqrt{6}\hbar Y_{20}, \hat{L}_- Y_{20} = \sqrt{6}\hbar Y_{2-1}, \hat{L}_- Y_{2-1} = 2\hbar Y_{2-2} \end{cases} \quad (126)$$

...

4) When $j = l$, from (57)-(59), we get

$$\begin{cases} \hat{L}_z Y_{ll} = l\hbar Y_{ll}, \hat{L}_+ Y_{ll} = 0; \\ \hat{L}_- Y_{ll} = \sqrt{2l}\hbar Y_{l-1}, \dots, \hat{L}_- Y_{l-1-l} = \sqrt{2l}\hbar Y_{l-l} \end{cases} \quad (127)$$

The following expressions can be deduced from (21)

$$\begin{cases} \hat{L}_+ = \hbar e^{i\varphi} \left(\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \varphi} \right) \\ \hat{L}_- = \hbar e^{-i\varphi} \left(-\frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \varphi} \right) \\ \hat{L}_z = -i\hbar \frac{\partial}{\partial \varphi} \end{cases} \quad (128)$$

Thus

1) When $j = 0$, the solution of the system (124) are

$$Y_{00} = \frac{1}{\sqrt{4\pi}} \quad (129)$$

2) When $j = 1$, the solution of the system (125) are

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\varphi}, Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta, Y_{1-1} = \sqrt{\frac{3}{8\pi}} \sin \theta e^{-i\varphi} \quad (130)$$

3) When $j = 2$, the solution of the system (126) are

$$Y_{2\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\varphi}, Y_{20} = \sqrt{\frac{5}{16\pi}} (3\cos^2 \theta - 1), Y_{2\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \cos \theta \sin \theta e^{\pm i\varphi} \quad (131)$$

...

4) When $j = l$, the solution of the system (127) are

$$Y_{ll} = \frac{(-1)^l}{2^l l!} \sqrt{\frac{(2l+1)!}{4\pi}} \sin^l \theta e^{il\varphi}, \dots, Y_{l-l} = \frac{1}{2^l l!} \sqrt{\frac{(2l+1)!}{4\pi}} \sin^l \theta e^{-il\varphi} \quad (132)$$

The spherical harmonic functions have been obtained by solving the first-order differential equations. Furthermore,

1) When $N = 2$, from (99), we get

$$\hat{A}_z \psi_{211} = 0, \psi_{200} = \frac{2}{k} \hat{A}_z \psi_{210} \quad (133)$$

2) When $N = 3$, from (113), we get

$$\hat{A}_z \psi_{322} = 0, \psi_{311} = \frac{3}{k} \hat{A}_z \psi_{321}, \psi_{300} = \frac{3\sqrt{6}}{4k} \hat{A}_z \psi_{310} - \frac{\sqrt{2}}{2} \psi_{320} \quad (134)$$

...

3) When $N = n$, (123) become

$$\left\{ \begin{array}{l} \hat{A}_z \psi_{nll} = 0, \psi_{nll-1} = \frac{k}{n} \sqrt{\frac{n^2-l^2}{2l+1}} \psi_{nl-1l-1}, \\ \psi_{nl-1l-2} = \frac{k}{n} \sqrt{\frac{n^2-(l-1)^2}{2l-1}} \psi_{nl-2l-2} + \frac{k}{n} \sqrt{\frac{n^2-l^2}{4l^2-1}} 4(l-1) \psi_{nll-2} \\ \cdots, \psi_{n10} = \frac{k}{n} \sqrt{\frac{n^2-1}{3}} \psi_{n00} + \frac{k}{n} \sqrt{\frac{n^2-4}{15}} 4 \psi_{n20} \end{array} \right. \quad (135)$$

It is well known that

$$\left\{ \begin{array}{l} \hat{p}_x = -i\hbar(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}) \\ \hat{p}_y = -i\hbar(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}) \\ \hat{p}_z = -i\hbar(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}) \end{array} \right. \quad (136)$$

Combining with (70), we have

$$\begin{aligned} \hat{A}_z = & \frac{\hbar^2}{2\mu} \left[\frac{1}{\hbar} e^{i\varphi} (\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial\theta} + \frac{i}{r\sin\theta} \frac{\partial}{\partial\varphi}) \hat{L}_- - \frac{1}{\hbar} e^{-i\varphi} (\sin\theta \frac{\partial}{\partial r} \right. \\ & \left. + \frac{\cos\theta}{r} \frac{\partial}{\partial\theta} - \frac{i}{r\sin\theta} \frac{\partial}{\partial\varphi}) \hat{L}_+ - 2(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial\theta}) \right] - k\cos\theta \end{aligned} \quad (137)$$

1) When $N=1$, we can deduce from (48) (80), (129) and (137)

$$R_{10} = \frac{2}{a^{\frac{3}{2}}} e^{-\frac{r}{a}} (a = \frac{\hbar^2}{\mu k}) \quad (138)$$

2) When $N=2$, we can deduce from (125), (129)-(130), (133) and (137)

$$R_{21} = \frac{1}{2\sqrt{6}a^{\frac{3}{2}}} \frac{r}{a} e^{-\frac{r}{2a}}, R_{20} = \frac{1}{\sqrt{2}a^{\frac{3}{2}}} (1 - \frac{r}{2a}) e^{-\frac{r}{2a}} \quad (139)$$

3) When $N=3$, we can deduce from (126), (129)-(131), (134) and (137)

$$\left\{ \begin{array}{l} R_{32} = \frac{4}{81\sqrt{30}a^{\frac{3}{2}}} (\frac{r}{a})^2 e^{-\frac{r}{3a}}, R_{20} = \frac{8}{27\sqrt{6}a^{\frac{3}{2}}} (1 - \frac{r}{6a}) \frac{r}{a} e^{-\frac{r}{3a}} \\ R_{30} = \frac{2}{3\sqrt{3}a^{\frac{3}{2}}} [1 - \frac{2r}{3a} + \frac{2}{27} (\frac{r}{a})^2] e^{-\frac{r}{3a}} \end{array} \right. \quad (140)$$

...

4) When $N=n$, we can deduce from (127), (129)-(131), ..., (132), (135) and (137)

$$\begin{cases} R_{nl} = \sqrt{\frac{2^{2l+3}}{(l+1)^{2l+3} a^{2l+3} \Gamma(2l+3)}} r^l e^{-\frac{r}{(l+1)a}}, \\ R_{nl-1} = \sqrt{\frac{2^{2l+1} l}{(l+1)^{2l+2} a^{2l+1} \Gamma(2l)}} \left[1 - \frac{r}{l(l+1)a}\right] r^{l-1} e^{-\frac{r}{(l+1)a}}, \\ \dots \end{cases} \quad (141)$$

Therefore, we are also able to obtain the radial functions by solving the first-order differential equations without the solution of the second order Schrodinger equation. These solutions are the same as the results obtained in wave mechanics on the hydrogen atom. As a result, the descriptions of matrix mechanics and wave mechanics on the hydrogen atom have been unified here.

It is well known that

$$[\hat{Q}_z, \hat{H}] = [\hat{Q}^2, \hat{H}] = [\hat{Q}_z, \hat{Q}^2] = 0$$

Hence the operators \hat{Q}_z, \hat{Q}^2 and \hat{H} have simultaneous eigenfunctions.

1) When $N=1$, we can get from (17), (48) and (78)-(80)

$$\hat{Q}_z \psi_{100} = \hat{Q}_+ \psi_{100} = \hat{Q}_z' \psi_{100} = \hat{Q}_+' \psi_{100} = \hat{Q}^2 \psi_{100} = 0 \quad (142)$$

2) When $N=2$, we can get from (17), (84), (95) and (99)

$$\begin{aligned} & \hat{Q}_z \begin{bmatrix} \psi_{200} & \psi_{21-1} & \psi_{210} & \psi_{211} \end{bmatrix} \\ &= \begin{bmatrix} \psi_{200} & \psi_{21-1} & \psi_{210} & \psi_{211} \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \quad (143)$$

The eigenvalues corresponding to the square matrix in (143) are

$$\lambda_1 = \lambda_2 = -\frac{\hbar}{2}, \lambda_3 = \lambda_4 = \frac{\hbar}{2} \quad (144)$$

According to Schurs theorem, there exists a unitary matrix in the following expression

$$\begin{bmatrix} \varphi_{2\frac{1}{2}-\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}-\frac{1}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \psi_{200} & \psi_{21-1} & \psi_{210} & \psi_{211} \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (145)$$

The square matrix in (143) is diagonalizable

$$\begin{aligned} & \hat{Q}_z \begin{bmatrix} \varphi_{2\frac{1}{2}-\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}-\frac{1}{2}} \end{bmatrix} \\ &= \begin{bmatrix} \varphi_{2\frac{1}{2}-\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}-\frac{1}{2}} \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \end{aligned} \quad (146)$$

Thus we can deduce from (17), (82)-(84), (95) and (97)-(99)

$$\begin{aligned} & \hat{Q}_+ \begin{bmatrix} \varphi_{2\frac{1}{2}-\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}-\frac{1}{2}} \end{bmatrix} \\ &= \begin{bmatrix} \varphi_{2\frac{1}{2}-\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}-\frac{1}{2}} \end{bmatrix} \hbar \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (147)$$

$$\begin{aligned} & \hat{Q}'_z \begin{bmatrix} \varphi_{2\frac{1}{2}-\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}-\frac{1}{2}} \end{bmatrix} \\ &= \begin{bmatrix} \varphi_{2\frac{1}{2}-\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}-\frac{1}{2}} \end{bmatrix} \frac{\hbar}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \end{aligned} \quad (148)$$

$$\begin{aligned} & \hat{Q}'_+ \begin{bmatrix} \varphi_{2\frac{1}{2}-\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}-\frac{1}{2}} \end{bmatrix} \\ &= \begin{bmatrix} \varphi_{2\frac{1}{2}-\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}-\frac{1}{2}} \end{bmatrix} \hbar \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (149)$$

$$\begin{aligned} & \hat{Q}^2 \begin{bmatrix} \varphi_{2\frac{1}{2}-\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}-\frac{1}{2}} \end{bmatrix} \\ &= \frac{3\hbar^2}{4} \begin{bmatrix} \varphi_{2\frac{1}{2}-\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}\frac{1}{2}} & \varphi_{2\frac{1}{2}-\frac{1}{2}} \end{bmatrix} \end{aligned} \quad (150)$$

3) When $N=3$, we can get from (17), (102), (109) and (113)

$$\begin{aligned}
& \hat{Q}_z [\psi_{300} \ \psi_{31-1} \ \psi_{310} \ \psi_{311} \ \psi_{322} \ \psi_{321} \ \psi_{320} \ \psi_{32-1} \ \psi_{32-2}] \\
&= [\psi_{300} \ \psi_{31-1} \ \psi_{310} \ \psi_{311} \ \psi_{322} \ \psi_{321} \ \psi_{320} \ \psi_{32-1} \ \psi_{32-2}] \\
& \quad \frac{\hbar}{2} \begin{bmatrix} 0 & 0 & \frac{2}{3}\sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{2}{3}\sqrt{6} & 0 & 0 & 0 & 0 & 0 & \frac{2}{3}\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3}\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix} \quad (151)
\end{aligned}$$

The eigenvalues corresponding to the square matrix in (151) are

$$\lambda_1 = \lambda_2 = \lambda_3 = -\hbar, \lambda_4 = \lambda_5 = \lambda_6 = 0, \lambda_7 = \lambda_8 = \lambda_9 = \hbar \quad (152)$$

According to Schurs theorem, there exists a unitary matrix in the following expression

$$\begin{aligned}
& [\varphi_{311} \ \varphi_{310} \ \varphi_{31-1} \ \phi_{31-1} \ \phi_{310} \ \phi_{311} \ \chi_{311} \ \chi_{310} \ \chi_{31-1}] \\
&= [\psi_{300} \ \psi_{31-1} \ \psi_{310} \ \psi_{311} \ \psi_{322} \ \psi_{321} \ \psi_{320} \ \psi_{32-1} \ \psi_{32-2}] \\
& \quad \frac{1}{6} \begin{bmatrix} 0 & 0 & 2\sqrt{3} & 0 & -2\sqrt{3} & 0 & 2\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & -3\sqrt{2} & 0 & 0 & 0 & 3\sqrt{2} & 0 \\ 0 & 0 & -3\sqrt{2} & 0 & 0 & 0 & 3\sqrt{2} & 0 & 0 \\ 0 & -3\sqrt{2} & 0 & 0 & 0 & 3\sqrt{2} & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3\sqrt{2} & 0 & 0 & 0 & 3\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2\sqrt{6} & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 3\sqrt{2} & 0 & 0 & 0 & 3\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix} \quad (153)
\end{aligned}$$

The square matrix in (151) is diagonalizable

$$\begin{aligned}
& \hat{Q}_z [\varphi_{311} \ \varphi_{310} \ \varphi_{31-1} \ \phi_{31-1} \ \phi_{310} \ \phi_{311} \ \chi_{311} \ \chi_{310} \ \chi_{31-1}] \\
&= [\varphi_{311} \ \varphi_{310} \ \varphi_{31-1} \ \phi_{31-1} \ \phi_{310} \ \phi_{311} \ \chi_{311} \ \chi_{310} \ \chi_{31-1}]
\end{aligned}$$

$$\hbar \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad (154)$$

Thus we can deduce from (17), (100)-(102), (109) and (111)-(113)

$$\begin{aligned} & \hat{Q}_+ [\varphi_{311} \quad \varphi_{310} \quad \varphi_{31-1} \quad \phi_{31-1} \quad \phi_{310} \quad \phi_{311} \quad \chi_{311} \quad \chi_{310} \quad \chi_{31-1}] \\ &= [\varphi_{311} \quad \varphi_{310} \quad \varphi_{31-1} \quad \phi_{31-1} \quad \phi_{310} \quad \phi_{311} \quad \chi_{311} \quad \chi_{310} \quad \chi_{31-1}] \\ & \quad \hbar \begin{bmatrix} 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (155)$$

$$\begin{aligned} & \hat{Q}'_z [\varphi_{311} \quad \varphi_{310} \quad \varphi_{31-1} \quad \phi_{31-1} \quad \phi_{310} \quad \phi_{311} \quad \chi_{311} \quad \chi_{310} \quad \chi_{31-1}] \\ &= [\varphi_{311} \quad \varphi_{310} \quad \varphi_{31-1} \quad \phi_{31-1} \quad \phi_{310} \quad \phi_{311} \quad \chi_{311} \quad \chi_{310} \quad \chi_{31-1}] \\ & \quad \hbar \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \end{aligned} \quad (156)$$

$$\hat{Q}'_+ [\varphi_{311} \quad \varphi_{310} \quad \varphi_{31-1} \quad \phi_{31-1} \quad \phi_{310} \quad \phi_{311} \quad \chi_{311} \quad \chi_{310} \quad \chi_{31-1}]$$

$$\begin{aligned}
&= [\varphi_{311} \quad \varphi_{310} \quad \varphi_{31-1} \quad \phi_{31-1} \quad \phi_{310} \quad \phi_{311} \quad \chi_{311} \quad \chi_{310} \quad \chi_{31-1}] \\
&\quad \hbar \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (157)
\end{aligned}$$

$$\begin{aligned}
&\hat{Q}^2 [\varphi_{311} \quad \varphi_{310} \quad \varphi_{31-1} \quad \phi_{31-1} \quad \phi_{310} \quad \phi_{311} \quad \chi_{311} \quad \chi_{310} \quad \chi_{31-1}] \\
&= 2\hbar^2 [\varphi_{311} \quad \varphi_{310} \quad \varphi_{31-1} \quad \phi_{31-1} \quad \phi_{310} \quad \phi_{311} \quad \chi_{311} \quad \chi_{310} \quad \chi_{31-1}] \quad (158)
\end{aligned}$$

...

Above, the specific matrix representations of the Lie algebra of the SU(2) are obtained by using the matrix transformations. We are also able to find

- 1) The eigenvalues of the Lie algebra of the group SU(2) is degenerate
- 2) The square matrices in (147), (149), (155) and (157) is not simultaneously able to become the Heisenberg matrix.
- 3) Let $\varphi_{\frac{1}{2}-\frac{1}{2}}, \varphi_{\frac{1}{2}\frac{1}{2}}$ be the orthonormalized simultaneous eigenfunctions of the operators \hat{Q}_z and \hat{Q}^2

$$\hat{Q}_z \varphi_{\frac{1}{2}-\frac{1}{2}} = -\frac{\hbar}{2} \varphi_{\frac{1}{2}-\frac{1}{2}}, \hat{Q}_z \varphi_{\frac{1}{2}\frac{1}{2}} = \frac{\hbar}{2} \varphi_{\frac{1}{2}\frac{1}{2}} \quad (159)$$

Let $\varphi'_{\frac{1}{2}-\frac{1}{2}}, \varphi'_{\frac{1}{2}\frac{1}{2}}$ be the orthonormalized simultaneous eigenfunctions of the operators \hat{Q}'_z and \hat{Q}'^2

$$\hat{Q}'_z \varphi'_{\frac{1}{2}-\frac{1}{2}} = -\frac{\hbar}{2} \varphi'_{\frac{1}{2}-\frac{1}{2}}, \hat{Q}'_z \varphi'_{\frac{1}{2}\frac{1}{2}} = \frac{\hbar}{2} \varphi'_{\frac{1}{2}\frac{1}{2}} \quad (160)$$

Thus

$$\begin{aligned}
&\hat{Q}_z [\varphi_{\frac{1}{2}-\frac{1}{2}} \varphi'_{\frac{1}{2}\frac{1}{2}} \quad \varphi_{\frac{1}{2}\frac{1}{2}} \varphi'_{\frac{1}{2}\frac{1}{2}} \quad \varphi_{\frac{1}{2}\frac{1}{2}} \varphi'_{\frac{1}{2}-\frac{1}{2}} \quad \varphi_{\frac{1}{2}-\frac{1}{2}} \varphi'_{\frac{1}{2}-\frac{1}{2}}] \\
&= [\varphi_{\frac{1}{2}-\frac{1}{2}} \varphi'_{\frac{1}{2}\frac{1}{2}} \quad \varphi_{\frac{1}{2}\frac{1}{2}} \varphi'_{\frac{1}{2}\frac{1}{2}} \quad \varphi_{\frac{1}{2}\frac{1}{2}} \varphi'_{\frac{1}{2}-\frac{1}{2}} \quad \varphi_{\frac{1}{2}-\frac{1}{2}} \varphi'_{\frac{1}{2}-\frac{1}{2}}] \\
&\quad \frac{\hbar}{2} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (161)
\end{aligned}$$

$$\begin{aligned}
& \hat{Q}'_z [\varphi_{\frac{1}{2}-\frac{1}{2}}\varphi'_{\frac{1}{2}\frac{1}{2}} \quad \varphi_{\frac{1}{2}\frac{1}{2}}\varphi'_{\frac{1}{2}\frac{1}{2}} \quad \varphi_{\frac{1}{2}\frac{1}{2}}\varphi'_{\frac{1}{2}-\frac{1}{2}} \quad \varphi_{\frac{1}{2}-\frac{1}{2}}\varphi'_{\frac{1}{2}-\frac{1}{2}}] \\
&= [\varphi_{\frac{1}{2}-\frac{1}{2}}\varphi'_{\frac{1}{2}\frac{1}{2}} \quad \varphi_{\frac{1}{2}\frac{1}{2}}\varphi'_{\frac{1}{2}\frac{1}{2}} \quad \varphi_{\frac{1}{2}\frac{1}{2}}\varphi'_{\frac{1}{2}-\frac{1}{2}} \quad \varphi_{\frac{1}{2}-\frac{1}{2}}\varphi'_{\frac{1}{2}-\frac{1}{2}}] \\
&\quad \frac{\hbar}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}
\end{aligned} \tag{162}$$

The above results of the group theory are different from (146) and (148). As a result, the descriptions of the group theory on the hydrogen atom have been explained.

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